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A note on the similarity solutions for free convection on a vertical plate

J.H. MERKIN

Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, UK

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Summary

The similarity solutions for free convection on a vertical plate when the (non-dimensional) plate temperature is x^{λ} and when the (non-dimensional) surface heat flux is $-x^{\mu}$ are considered. Solutions valid for $\lambda \gg 1$ and $\mu \gg 1$ are obtained. Further, for the first problem it is shown that there is a value λ_0, dependent on the Prandtl number, such that solutions of the similarity equations are possible only for $\lambda > \lambda_0$, and for the second problem that solutions are possible only for $\mu > -1$ (for all Prandtl numbers). In both cases the solutions becomes singular as $\lambda \to \lambda_0$ and as $\mu \to -1$, and the natures of these singularities are discussed.

1. Introduction

The problem of obtaining similarity solutions for the free convection boundary-layer equations governing the flow on a heated vertical plate was first considered by Sparrow and Gregg [1]. They showed that when the (non-dimensional) plate temperature T_w was given by $T_w = x^{\lambda}$(λ a constant) the governing partial differential equations could be reduced to a pair of coupled ordinary differential equations by a suitable change of variables. They gave results for values of λ between -0.8 and 3.0. The case $\lambda = 0$ corresponds to a uniform plate temperature and has been treated separately by Pohlhausen [2] and Ostrach [3]. As well as being of interest in themselves [4,5], these similarity equations arise as the leading-order solutions in series expansions in problems where other effects are present, for example in mixed convection [6,7,8,9,10,11] and in magnetohydrodynamic free convection [12,13]. Consequently it is important to have a full understanding of their solution.

The purpose of this paper is twofold. We first obtain a solution when $\lambda \gg 1$; here the leading-order term is given by the similarity solution for a plate temperature $T_w = e^{mx}$ (m a constant) given by [1]. We then show, by a numerical integration of the equations, there is a value of $\lambda = \lambda_0$ (say) such that a solution is possible only for $\lambda > \lambda_0$, where λ_0 depends on the Prandl number P_r. The solution becomes singular as $\lambda \to \lambda_0$, and we describe the nature of this singularity.

We then go on to consider the case when the plate is heated in a prescribed way. Again a similarity transformation is possible when the normal gradient of temperature on the plate $(\partial T/\partial y)_{y=0} = -x^{\mu}$ (μ a constant). The case when $\mu = 0$ has been given by Sparrow and Gregg [14]. A solution for $\mu \gg 1$ is obtained, analogous to the prescribed temperature case. Again there is a lower bound on μ for solutions to the similarity equations to exist. However, we show by a simple argument that this bound is now independent of P_r. In fact we find we need $\mu > -1$ with the solution becoming singular as $\mu \to -1$.

The scheme of the paper is as follows. We describe in some detail the solution for the prescribed temperature case. We then sketch briefly the corresponding solution for the prescribed heating case, highlighting, where necessary, the differences between the two cases.

2. Prescribed plate temperature

The (non-dimensional) equations for the boundary-layer flow on a heated vertical plate are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = T + \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where x measures distance along the plate and y normal to it; u and v are the velocity components in the x and y directions respectively and T is the temperature difference. The boundary conditions for this case are

$$\begin{aligned} &u = v = 0, \quad T = T_w = x^\lambda \quad &&\text{on } y = 0, \\ &u \to 0, \quad T \to 0 &&\text{as } y \to \infty. \end{aligned} \tag{4}$$

The systems (1)–(4) can be reduced to similarity form by the transformation

$$\psi = x^{3+\lambda/4} f(\eta), \quad T = x^\lambda \theta(\eta), \quad \eta = y x^{\lambda - 1/4} \tag{5}$$

where ψ is the stream function defined in the usual way. Using (5), equations (1)–(3) become

$$f''' + \theta + \left(\frac{3+\lambda}{4}\right) f f'' - \left(\frac{1+\lambda}{2}\right) f'^2 = 0, \tag{6}$$

$$\frac{1}{P_r}\theta'' + \left(\frac{3+\lambda}{4}\right) f\theta' - \lambda f'\theta = 0 \tag{7}$$

with (4) giving the boundary conditions

$$\begin{aligned} &f = f' = 0, \quad \theta = 1 \quad &&\text{on } \eta = 0, \\ &f' \to 0, \quad \theta \to 0 &&\text{as } \eta \to \infty \end{aligned} \tag{8}$$

(here dashes denote differentiation with respect to η). Equations (6) and (7) are essentially the equations given in [1].

To find a solution of (6), (7) and (8) valid when $\lambda \gg 1$, we make the further transformation

$$f=\lambda^{-3/4}F(\bar{\eta}), \quad \theta=\theta(\bar{\eta}), \quad \bar{\eta}=\lambda^{1/4}\eta. \tag{9}$$

On substituting (9) into (6) and (7) we obtain

$$F''' + \theta + \tfrac{1}{4}\left(1+\frac{3}{\lambda}\right)FF'' - \tfrac{1}{2}\left(1+\frac{1}{\lambda}\right)F'^2 = 0, \tag{10}$$

$$\frac{1}{\mathrm{P_r}}\theta'' + \tfrac{1}{4}\left(1+\frac{3}{\lambda}\right)F\theta' - \theta F' = 0 \tag{11}$$

(dashes now denote differentiation with respect to $\bar{\eta}$). The boundary conditions to be satisfied are still given by (8). We look for a solution of (10) and (11) by expanding F and θ in the form

$$\begin{aligned} F &= F_0 + \lambda^{-1}F_1 + \ldots, \\ \theta &= \theta_0 + \lambda^{-1}\theta_1 + \ldots. \end{aligned} \tag{12}$$

F_0 and θ_0 satisfy the equations

$$F_0''' + \theta_0 + \tfrac{1}{4}F_0F_0'' - \tfrac{1}{2}F_0'^2 = 0, \tag{13}$$

$$\frac{1}{\mathrm{P_r}}\theta_0'' + \tfrac{1}{4}F_0\theta_0' - F_0'\theta_0 = 0, \tag{14}$$

with boundary conditions

$$\begin{aligned} &F_0 = F_0' = 0, \quad \theta_0 = 1 \qquad \text{on } \bar{\eta} = 0, \\ &F_0' \to 0, \quad \theta_0 \to 0 \qquad \text{as } \bar{\eta} \to \infty. \end{aligned} \tag{15}$$

Equations (13) and (14) are the equations for the case when $T_w = \mathrm{e}^{mx}$, [1]. The equations for the higher-order terms in (12) are all linear and can be solved numerically in a straightforward way once the solution of (13) and (14) is known. We find, for $\mathrm{P_r} = 1$, that

$$\begin{aligned} \left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)_0 &= \lambda^{-1/4}(0.8515 - 0.1579\lambda^{-1} + \ldots), \\ \left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)_0 &= -\lambda^{1/4}(0.5823 - 0.0009\lambda^{-1} + \ldots). \end{aligned} \tag{16}$$

Values of $(\mathrm{d}^2 f/\mathrm{d}\eta^2)_0$ and $(\mathrm{d}\theta/\mathrm{d}\eta)_0$ obtained by solving equations (6) and (7) numerically are compared with their values calculated using (16) in Table 1. We can see that the two are in good agreement even at modest values of λ. At $\lambda = 4$, $(\mathrm{d}^2 f/\mathrm{d}\eta^2)_0$ is 0.5% in error, while the error in $(\mathrm{d}\theta/\mathrm{d}\eta)_0$ is only 0.2%.

Table 1. Values of $f''(0)$ and $\theta'(0)$ obtained from (16) and by a numerical integration of equations (6) and (7)

λ	$f''(0)$		$\theta'(0)$	
	Exact	Series (16)	Exact	Series (16)
1.00	0.7395	0.6936	−0.5951	−0.5814
1.25	0.7155	0.6858	−0.6251	−0.6150
1.50	0.6949	0.6743	−0.6516	−0.6438
1.75	0.6769	0.6619	−0.6754	−0.6692
2.00	0.6611	0.6496	−0.6970	−0.6920
2.25	0.6469	0.6379	−0.7169	−0.7132
2.50	0.6341	0.6269	−0.7354	−0.7318
2.75	0.6225	0.6166	−0.7526	−0.7495
3.00	0.6119	0.6070	−0.7687	−0.7660
3.25	0.6021	0.5980	−0.7839	−0.7815
3.50	0.5931	0.5895	−0.7982	−0.7961
3.75	0.5847	0.5816	−0.8119	−0.8100
4.00	0.5768	0.5742	−0.8249	−0.8232

Next consider the behaviour of the solution for $\lambda < 0$. As λ is decreased from $\lambda = 0$, the thickness of the boundary layer (in terms of η) decreases, while $(d^2f/d\eta^2)_0$ increases. $(d\theta/d\eta)_0$ changes sign (from negative to positive) at $\lambda = -3/5$. These effects become more pronounced as λ is decreased further and the solution appears to be becoming singular as λ approaches a value λ_0 (say). This can be clearly seen in Figure 1, where $f''(0)$ and $\theta'(0)$ are plotted against λ, and also in Figure 2 where temperature profiles $\theta(\eta)$ are given for various λ close to λ_0.

To obtain the solution near this singularity we put $\lambda = \lambda_0 + \epsilon$ where $\epsilon \ll 1$ and then make the transformation

$$f = \epsilon^{-1/4}\phi(z), \quad \theta = \epsilon^{-1}H(z), \quad z = \epsilon^{-1/4}\eta. \tag{17}$$

The reason for this transformation will become apparent later. Substituting (17) into equations (6) and (7) gives

$$\phi''' + H + \tfrac{1}{4}(3 + \lambda_0 + \epsilon)\phi\phi'' - \tfrac{1}{2}(1 + \lambda_0 + \epsilon)\phi'^2 = 0, \tag{18}$$

$$\frac{1}{P_r}H'' + \tfrac{1}{4}(3 + \lambda_0 + \epsilon)\phi H' - (\lambda_0 + \epsilon)\phi' H = 0, \tag{19}$$

with (8) giving the boundary conditions

$$\begin{aligned} &\phi = \phi' = 0, \quad H = \epsilon \quad \text{on } z = 0, \\ &\phi' \to 0, \quad H \to 0 \quad \text{as } z \to \infty \end{aligned} \tag{20}$$

(dashes denote differentiation with respect to z). The forms of (18), (19) and (20) suggest an expansion in the form

$$\begin{aligned} \phi &= \phi_0 + \epsilon\phi_1 + \ldots, \\ H &= H_0 + \epsilon H_1 + \ldots. \end{aligned} \tag{21}$$

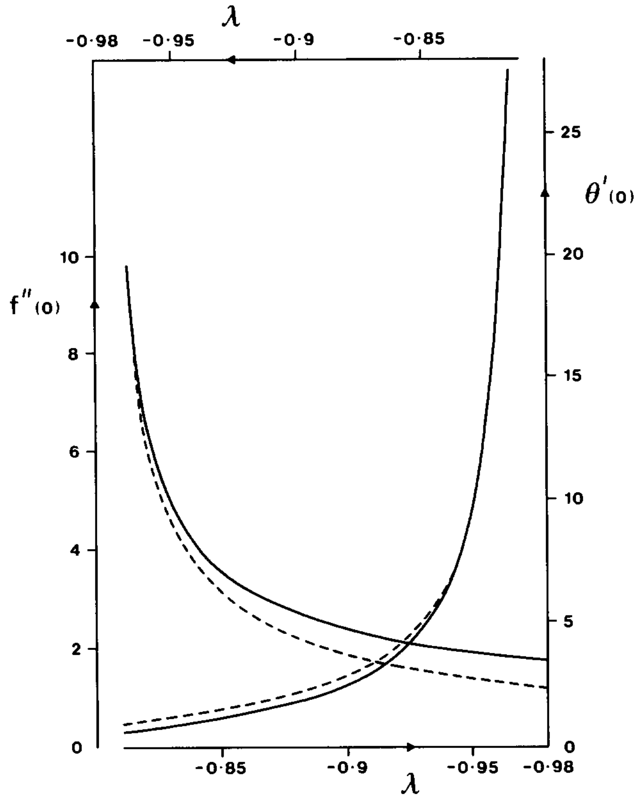

Figure 1. $\theta'(0)$ and $f''(0)$ plotted against λ for λ near λ_0 (the values given by (33) are shown by the broken lines).

The equations for leading-order terms are

$$\phi_0''' + H_0 + \tfrac{1}{4}(3+\lambda_0)\phi_0\phi_0'' - \tfrac{1}{2}(1+\lambda_0)\phi_0'^2 = 0, \tag{22}$$

$$\frac{1}{\mathrm{P_r}} H_0'' + \tfrac{1}{4}(3+\lambda_0)\phi_0 H_0' - \lambda_0\phi_0' H_0 = 0, \tag{23}$$

with boundary conditions

$$\begin{aligned} &\phi_0 = \phi_0' = 0 \qquad \text{on } z=0,\\ &\phi_0' \to 0, \quad H_0 \to 0 \qquad \text{as } z\to\infty. \end{aligned} \tag{24}$$

The homogeneous system given by (22)–(24) is an eigenvalue problem for λ_0. This problem has arisen previously but in a different context, [15], (though the equations given in [15] are different to (22) and (23) they can be transformed into them by a simple change of variables). We find that λ_0 is dependent on $\mathrm{P_r}$ and values of λ_0 for various $\mathrm{P_r}$ are given in Table 2 (in particular we find, for $\mathrm{P_r} = 1$, that $\lambda_0 = -0.9790$).

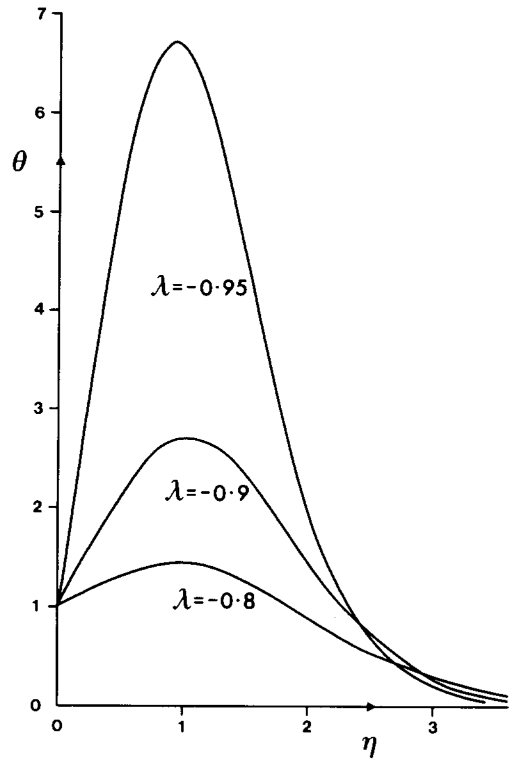

Figure 2. Temperature profiles θ plotted against η for $\lambda = -0.8$, $\lambda = -0.9$, $\lambda = -0.95$ and $P_r = 1$.

The solution is not unique, and we fix our solution by taking $\phi_0''(0) = 1$. Values of the corresponding $H_0'(0)$ are also given in Table 2. The actual solution will, in general, not have $\phi_0''(0) = 1$, but will have $\phi_0''(0) = C$ (say) for some constant C to be determined. The general leading-order solution $(\bar{\phi}_0, \bar{H}_0)$ can be then obtained from our basic solution (ϕ_0, H_0) by writing

$$\bar{\phi}_0 = C^{1/3}\phi_0, \quad \bar{H}_0 = C^{4/3}H_0, \quad \bar{z} = C^{1/3}z. \tag{25}$$

The value of C is determined by the solution of the equations of $O(\epsilon)$, which we now consider. These are

$$\bar{\phi}_1''' + H_1 + \tfrac{1}{4}(3+\lambda_0)(\bar{\phi}_0\bar{\phi}_1'' + \bar{\phi}_1\bar{\phi}_0'') - (1+\lambda_0)\bar{\phi}_0'\bar{\phi}_1'$$

$$= C^{4/3}\left(\tfrac{1}{2}\bar{\phi}_0'^2 - \tfrac{1}{4}\bar{\phi}_0\bar{\phi}_0''\right), \tag{26}$$

$$\frac{1}{P_r}H_1'' + \tfrac{1}{4}(3+\lambda_0)(\bar{\phi}_0 H_1' + \bar{\phi}_1\bar{H}_0') - \lambda_0(\bar{\phi}_0' H_1 + \bar{\phi}_1'\bar{H}_0)$$

$$= C^{4/3}\left(\bar{\phi}_0'\bar{H}_0 - \tfrac{1}{4}\bar{\phi}_0\bar{H}_0'\right), \tag{27}$$

Table 2. Values of the eigenvalues λ_0 (and corresponding $H_0'(0)$ of equations (22) and (23) for various values of P_r

P_r	λ_0	$H_0'(0)$
0.2	−1.1690	0.3044
0.4	−1.0606	0.4747
0.6	−1.0204	0.5930
0.7	−1.0070	0.6433
0.8	−0.9960	0.6895
1.0	−0.9790	0.7729
1.2	−0.9664	0.8476
1.4	−0.9566	0.9160
1.6	−0.9487	0.9794
1.8	−0.9422	1.0391
2.0	−0.9368	1.0955
2.5	−0.9263	1.2259
3.0	−0.9188	1.3448
4.0	−0.9086	1.5588
5.0	−0.9019	1.7501
6.0	−0.8971	1.9277
8.0	−0.8907	2.2492
10.0	−0.8865	2.5403

with boundary conditions

$$\begin{aligned}&\bar{\phi}_1 = \bar{\phi}_1' = 0, \quad H_1 = 1 \qquad \text{on } \bar{z} = 0,\\ &\bar{\phi}_1' \to 0, \quad H_1 \to 0 \qquad \text{as } \bar{z} \to \infty.\end{aligned} \tag{28}$$

Here we have applied the transformation (25) and written $\phi_1 = C^{-1}\bar{\phi}_1$.

To solve equations (26) and (27) numerically we construct four separate solutions, namely two complementary functions (ϕ_a, H_a) and (ϕ_b, H_b) (with $\phi_a''(0) = 1$, $H_a'(0) = 0$ and $\phi_b''(0) = 0$, $H_b'(0) = 1$) and two particular integrals (ϕ_c, H_c) (which is a solution of (26) and (27) with the right-hand sides put to zero but with $\phi_c''(0) = 0$, $H_c(0) = 1$ and $H_c'(0) = 0$) and (ϕ_d, H_d) (which is a solution of the full equations with $C^{4/3}$ replaced by 1 and $\phi_d''(0) = H_d(0) = H_d'(0) = 0$). The complete solution is then

$$\begin{aligned}&\bar{\phi}_1 = \alpha\phi_a + \beta\phi_b + \phi_c + C^{4/3}\phi_d,\\ &H_1 = \alpha H_a + \beta H_b + H_c + C^{4/3}H_d.\end{aligned} \tag{29}$$

As $\bar{z} \to \infty$, we have, from (26) and (27), that $H_i \to A_i$ and $\phi_i' \sim -A_i\bar{z} + B_i$ $(i = a, b, c, d)$, where the A_i and B_i are constants. So to satisfy (28) we must chose α and β so that

$$\begin{aligned}&\alpha A_a + \beta A_b + A_c + C^{4/3}A_d = 0,\\ &\alpha B_a + \beta B_b + B_c + C^{4/3}B_d = 0.\end{aligned} \tag{30}$$

Now, equations (26) and (27) possess a complementary function (ϕ_1^*, H_1^*), where $\phi_1^* = \bar{z}\bar{\phi}_0' + \bar{\phi}_0$, $H_1^* = \bar{z}\bar{H}_0' + 4\bar{H}_0$, which satisfies the homogeneous boundary conditions $\phi_1^* = \phi_1^{*\prime} = H_1^* = 0$ on $\bar{z} = 0$, $\phi_1^{*\prime} \to 0$, $H_1^* \to 0$ as $\bar{z} \to \infty$. This, in turn, implies that the equations

$\alpha A_a + \beta A_b = 0$, $\alpha B_a + \beta B_b = 0$ must have non-trivial solutions, and, in particular,

$$A_a B_b - A_b B_a = 0. \tag{31}$$

Hence equations (30) cannot be solved for α and β, and are, in fact, a pair of equations which determine C. We find, after a little manipulation, and using (31), that

$$C^{4/3} = \frac{A_b B_c - A_c B_b}{A_d B_b - A_b B_d}. \tag{32}$$

From numerical integrations with $\mathrm{P_r} = 1$, we find that $C = 0.31943$.

We can now see why the transformation (17) was made. Any other transformation would lead us to have equations with either $H_1(0) = 0$ with the same terms on the right-hand sides of (26) and (27) or $H_1(0) = 1$ and with zero on the right-hand sides at the first perturbation from leading order. The solution would then require only three of the four integrations described above, namely (ϕ_a, H_a), (ϕ_b, H_b) and either (ϕ_c, H_c) or (ϕ_d, H_d). Consequently there would be only three terms in the equations corresponding to (30) required to satisfy the outer boundary conditions. Then because of the existence of the complementary function (ϕ_1^*, H_1^*) (and, in particular (31)) these could not be solved. The only way out of this difficulty is to make transformation (17) so that both the boundary condition on H and the perturbation from λ_0 contribute to the $O(\epsilon)$ equations. This in turn fixes the value of C.

Finally, we have, for $\mathrm{P_r} = 1$,

$$\begin{aligned} \left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)_0 &= C(\lambda - \lambda_0)^{-3/4} + \dots, \\ \left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right)_0 &= 0.7729C(\lambda - \lambda_0)^{-5/4} + \dots \end{aligned} \tag{33}$$

as $\lambda \to \lambda_0$. Values of $(\mathrm{d}^2 f/\mathrm{d}\eta^2)_0$ and $(\mathrm{d}\theta/\mathrm{d}\eta)_0$ as given by (33) are also shown in Figure 1 (by the broken lines) and confirm the above theory.

3. Prescribed plate heating

Equations (1)–(3) also reduce to similarity form when the normal gradient of temperature on the plate is prescribed by $(\partial T/\partial y)_{y=0} = -x^{\mu}$ (μ a constant). To do this we put

$$\psi = x^{4+\mu/5}\chi(\zeta), \quad T = x^{1+4\mu/5} g(\zeta), \quad \zeta = y x^{\mu - 1/5}. \tag{34}$$

Equations (1)–(3) give

$$\chi''' + g + \left(\frac{4+\mu}{5}\right)\chi\chi' - \left(\frac{3+2\mu}{5}\right)\chi'^2 = 0, \tag{35}$$

$$\frac{1}{\mathrm{P_r}} g'' + \left(\frac{4+\mu}{5}\right)\chi g' - \left(\frac{1+4\mu}{5}\right)\chi' g = 0, \tag{36}$$

Table 3. Values of $\chi''(0)$ and $g(0)$ obtained from (38) and by a numerical integration of equations (35) and (36)

μ	$\chi''(0)$		$g(0)$	
	Exact	Series (38)	Exact	Series (38)
1.00	1.0097	0.8621	1.5148	1.5335
1.25	0.9597	0.8663	1.4671	1.4815
1.50	0.9174	0.8536	1.4268	1.4380
1.75	0.8809	0.8350	1.3920	1.4010
2.00	0.8491	0.8146	1.3617	1.3690
2.25	0.8209	0.7942	1.3347	1.3408
2.50	0.7957	0.7746	1.3106	1.3157
2.75	0.7729	0.7559	1.2897	1.2932
3.00	0.7524	0.7384	1.2699	1.2728
3.25	0.7336	0.7219	1.2517	1.2541
3.50	0.7164	0.7065	1.2349	1.2370
3.75	0.7005	0.6921	1.2194	1.2212
4.00	0.6858	0.6785	1.2050	1.2065
4.25	0.6721	0.6657	1.1915	1.1929
4.50	0.6594	0.6537	1.1789	1.1801
4.75	0.6474	0.6424	1.1670	1.1680
5.00	0.6361	0.6317	1.1558	1.1567

with boundary conditions

$$\begin{aligned} &\chi = \chi' = 0, \quad g' = -1 \qquad \text{on } \zeta = 0, \\ &\chi' \to 0, \quad g \to 0 \qquad \text{as } \zeta \to \infty. \end{aligned} \tag{37}$$

(now dashes denote differentiation with respect to ζ). The case with $\mu = 0$ was given by [14].

A solution valid for $\mu \gg 1$ can be obtained by first putting $\chi = \mu^{-4/5}\bar{\chi}$, $g = \mu^{-1/5}\bar{g}$ and $\bar{\zeta} = \mu^{1/5}\zeta$ and then looking for a solution of the resulting equations in descending powers of μ. The process is straightforward and follows closely the prescribed temperature case. Again the leading-order equations correspond to the similarity solution with the prescribed heat flux $(\partial T/\partial y)_{y=0} = -e^{mx}$. We find, for $P_r = 1$ that

$$\begin{aligned} \chi''(0) &= \mu^{-2/5}(1.2878 - 0.4257\mu^{-1} + \ldots), \\ g(0) &= \mu^{-1/5}(1.6116 - 0.0780\mu^{-1} + \ldots). \end{aligned} \tag{38}$$

Values of $\chi''(0)$ and $g(0)$ obtained from (38) are given in Table 3, together with their values obtained from a numerical integration of equations (35) and (36). As before, the two sets of values are in good agreement even for modest values of μ.

The situation for $\mu < 0$ is slightly different to the prescribed temperature case. Again, as μ is decreased from $\mu = 0$, the thickness of the layer decreases and both $\chi''(0)$ and $g(0)$ increase, and appear to approach a singularity as $\mu \to -1$. This can be seen in Figure 3 where graphs of $\chi''(0)$ and $g(0)$ are plotted against μ.

We can show directly that equations (35) and (36) cannot have a solution when $\mu \leqslant -1$. By integrating equation (36) once, and applying (37) we can show that

$$P_r(1 + \mu)\int_0^\infty \chi' g \, d\zeta = 1. \tag{39}$$

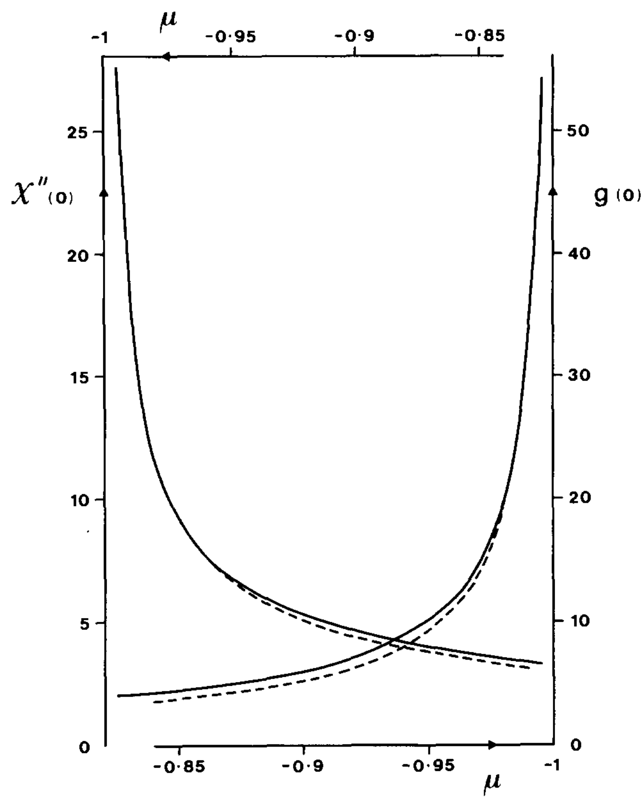

Figure 3. $g(0)$ and $\chi''(0)$ plotted against μ for μ near -1 (the values given by (51) are shown by the broken lines).

Clearly (39) shows that there can be no solution in which $\chi' > 0$ and $g > 0$ for all ζ when $\mu < -1$. Also we can show that equations (35) and (36) cannot have a solution when $\mu = -1$. For, with $\mu = -1$, equation (36) can be integrated once to give

$$g' + \tfrac{3}{5}\mathrm{P_r}\chi g = A \tag{40}$$

where A is a constant. Clearly A cannot be chosen to be compatible with the boundary conditions on both $\zeta = 0$ and as $\zeta \to \infty$.

To discuss the nature of the singularity near $\mu = -1$, we put $\mu = -1 + \delta$, where $\delta \ll 1$, and then make the transformation

$$\chi = \delta^{-1/5}X, \quad g = \delta^{-4/5}G, \quad \tau = \delta^{-1/5}\zeta. \tag{41}$$

As before this transformation is motivated by the fact that we require the perturbation of μ from $\mu = -1$ and that arising from the boundary condition on g to both contribute to the equations for the first perturbation from leading order.

Equation (41) is substituted into (35) and (36) and a solution of the resulting equations is sought by expanding in the form:

$$
\begin{aligned}
X &= X_0 + \delta X_1 + \dots, \\
G &= G_0 + \delta G_1 + \dots .
\end{aligned}
\tag{42}
$$

The equations for the leading-order terms are

$$X_0''' + G_0 + \tfrac{3}{5} X_0 X_0'' - \tfrac{1}{5} X_0'^2 = 0, \tag{43}$$

$$\frac{1}{\mathrm{P_r}} G_0' + \tfrac{3}{5} X_0 G_0 = 0, \tag{44}$$

subject to the boundary conditions

$$
\begin{aligned}
&X_0 = X_0' = G_0' = 0 \qquad \text{on } \tau = 0, \\
&X_0' \to 0, \quad G_0 \to 0 \qquad \text{as } \tau \to \infty .
\end{aligned}
\tag{45}
$$

(dashes denote differentiation with respect to τ.) To obtain equation (44) we have integrated the equation corresponding to (36) once; the boundary conditions on $\tau = 0$ and as $\tau \to \infty$ are now compatible.

The solution of (43) and (44) is not unique. To fix a solution we take $X_0''(0) = 1$, this gives, for $\mathrm{P_r} = 1$, $G_0(0) = 0.7389$. The solution with $X_0''(0) = D$ can then be obtained from this solution by putting

$$\overline{X}_0 = D^{1/3} X_0, \quad \overline{G}_0 = D^{4/3} G_0, \quad \bar{\tau} = D^{1/3} \tau. \tag{46}$$

The equations for the terms of $O(\delta)$ are, on putting $\overline{X}_1 = D^{4/3} X_1$, $\overline{G}_1 = D^{1/3} G_1$,

$$\overline{X}_1''' + \overline{G}_1 + \tfrac{3}{5}\left(\overline{X}_0 \overline{X}_1'' + \overline{X}_1 \overline{X}_0''\right) - \tfrac{2}{5} \overline{X}_0' \overline{X}_1' = D^{5/3}\left(\tfrac{2}{5} \overline{X}_0'^2 - \tfrac{1}{5} \overline{X}_0 \overline{X}_0''\right), \tag{47}$$

$$\frac{1}{\mathrm{P_r}} \overline{G}_1'' + \tfrac{3}{5}\left(\overline{X}_0 \overline{G}_1' + \overline{X}_1 \overline{G}_0' + \overline{X}_0' \overline{G}_1 + \overline{X}_1' \overline{G}_0\right) = D^{5/3}\left(\tfrac{4}{5} \overline{X}_0' G_0 - \tfrac{1}{5} \overline{X}_0 \overline{G}_0'\right), \tag{48}$$

with boundary conditions

$$
\begin{aligned}
&\overline{X}_1 = \overline{X}_1' = 0, \quad \overline{G}_1' = -1 \qquad \text{on } \bar{\tau} = 0, \\
&\overline{X}_1' \to 0, \quad \overline{G}_1 \to 0 \qquad \text{as } \bar{\tau} \to \infty .
\end{aligned}
\tag{49}
$$

We can show, by integrating (48) once, that any integration of equations (47) and (48) to find a complementary function (with $\bar{g}_1'(0) = 0$) has the property that $\overline{G}_1 \to 0$ as $\bar{\tau} \to \infty$. However, this is not the case for the particular integrals (X_c, G_c) and (X_d, G_d), where $G_c'(0) = -1$ with the right-hand sides of (47) and (48) put to zero, and $G_d'(0) = 0$ with D put to unity in (47) and (48). Both these solutions have the property that $G_c \to E_c$, $G_d \to E_d$ as $\bar{\tau} \to \infty$ (for some constants E_c and E_d). So the combination of complementary functions and particular integrals (as described above in the previous section) will have, as $\bar{\tau} \to \infty$,

$$G_1 \to E_c + D^{5/3} E_d. \tag{50}$$

Hence D must be chosen so that $D^{5/3} = -E_c/E_d$. The numerical integrations for $\mathrm{P_r} = 1$ give $D = 1.0899$. Then near $\mu = -1$,

$$\begin{aligned} \chi''(0) &= D(\mu+1)^{-3/5} + \ldots, \\ g(0) &= 0.7389 D^{4/3} (\mu+1)^{-4/5} + \ldots. \end{aligned} \tag{51}$$

Values of $\chi''(0)$ and $g(0)$ obtained from (51) are also shown in Figure 3.

4. Discussion

We have shown that for the prescribed plate-temperature problem a solution exists only for $\lambda > \lambda_0$, with λ_0 dependent on Prandl number. Here we consider the value of λ_0 in the limiting cases of small and large $\mathrm{P_r}$.

For small $\mathrm{P_r}$, as shown by Kuiken [16], there is an inner layer of thickness $O(1)$, in which the temperature is constant, governed by a Falkner-Skan-type equation [17,18] with Falkner-Skan parameter $\beta = 2(1+\lambda)/(3+\lambda)$ i.e. $\lambda = (3\beta - 2)/(2-\beta)$. Now, from [18], solutions of this equation exist only for $\beta > -0.1988$ suggesting a value of $\lambda_0 = -1.1808$ as $\mathrm{P_r} \to 0$, which is not inconsistent with the values of λ_0 given in Table 2.

For large $\mathrm{P_r}$, there is, [19], an inner region of thickness $O(\mathrm{P_r}^{-1/4})$ next to the plate, governed by the equations

$$f''' + \theta = 0, \tag{52}$$

$$\theta'' + \tfrac{1}{4}(3+\lambda) f\theta' - \lambda f'\theta = 0, \tag{53}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad \theta(0) = 1, \quad f''(\infty) = 0, \quad \theta(\infty) = 0 \tag{54}$$

(after rescaling (6) and (7) as suggested by [19]). The limiting value λ_0 for the existence of solutions of (52) and (53) has to be determined numerically, and we find a value of $\lambda_0 = 0.856$, again not inconsistent with the values shown in Table 2.

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